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Can p-Adic Numbers be Useful to Regularize Divergent Expectation Values of Quantum Observables?

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We show how p-adic analysis can be used in some cases to treat divergent series in quantum mechanics. We consider examples in which the usual theory of the Schrödinger equation would give rise to an infinite expectation value of the energy operator. By using p-adic analysis, we are able to get a convergent expansion and obtain a finite rational value for the energy. We present also the main ideas to interpret a quantum mechanical state by means of p-adic statistics.

1. INTRODUCTION

The first quantum model over the field of p-adic numbers $\mathbf{Q}_p$ was considered by Beltrametti and Cassinelli (1962), investigating the problem of the choice of a number field in quantum theories from the position of quantum logic. Great interest in p-adic physics has been present in some researches in string theory (Volovich, 1987, 1988; Grossman, 1987; Freund and Olson, 1987; Olson *et al.*, 1987; Frampton and Okada, 1988).

The main idea of these p-adic string investigations was to attempt to describe the space-time at Planck distances with the aid of the field of p-adic numbers $\mathbf{Q}_p$. This agrees with an old idea about violations of Archimedean axioms at Planck distances and the non-Archimedean number field $\mathbf{Q}_p$ can be a good mathematical object to describe such physical models.

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In these articles problems with the physical interpretation of such high-level models as p-adic strings were presented and so simpler models such as p-adic quantum mechanics and field theory were also investigated (Vladimirov and Volovich, 1989*a,b*; Volovich *et al.*, 1990; Khrennikov, 1990, 1991).

There are two main approaches to p-adic quantization. The first approach is based on a complex-valued wave function of p-adic argument $\psi: \mathbf{Q}_p^3 \rightarrow \mathbf{C}$.

The second one is based on a wave function of p-adic argument which assumes its value in some extensions of $\mathbf{Q}_p$ such as the quadratic extensions or the field $\mathbf{C}_p$ of complex p-adic number; for the definition of this field see, for instance, Mahler (1973).

We are interested in the second approach; the reader can find a description of the first approach in [Vladimir and Volovich (1989*a,b*), Volovich *et al.* (1990), and Olson *et al.* (1987) and a description of the second one in Khrennikov (1990, 1991).

In this paper we shall use p-adic numbers to describe quantum mechanical models to compute the mean energy value. Our main idea is the following.

We interpret the symbols x_1, x_2, x_3 used for the coordinates in the Schrödinger equation like formal variables and the wave amplitude $\psi(x_1, x_2, x_3)$ as a formal series:

$$\psi(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{nmk} x_1^n x_2^m x_3^k$$

The coefficients f_{nm} of the expansion are assumed to be rational numbers; if one assumes that the variables x_1, x_2, x_3 are real, one gets a standard wave function ψ; on the contrary, by assuming that the variables x_1, x_2, x_3 are p-adic numbers, one gets a new model of quantum mechanics.

Let $\psi(x)$ (x real) be a map not belonging to $L_2(\mathbf{R}, dx)$. In some cases, its p-adic realization ψ (x p-adic) belongs to $L_2(\mathbf{Q}_p, dx)$; see Khrennikov (1990) for p-adic Lebesgue distributions.

In such cases there is the opportunity to interpret $\psi(x)$ as the p-adic-valued quantum mechanical wave function and so consider new models in quantum mechanics.

It is also possible that ψ belongs to both functional spaces $L_2(\mathbf{R}, dx)$, $L_2(\mathbf{Q}_p, dx)$; in this case, however, the normalization constants are in general different, since the values of the integrals which one has to perform are completely different. Namely, if

$$A = \|\psi\|_{L_2(\mathbf{R}, dx)} \in \mathbf{R} \qquad \text{and} \qquad B = \|\psi\|_{L_2(\mathbf{Q}_p, dx)} \in \mathbf{Q}_p$$

we denote by $\psi_{n,\mathbf{R}} = \psi/\sqrt{A}$ and $\psi_{n,\mathbf{Q}_p} = \psi/\sqrt{B}$ the normalized states, respectively.

Let us consider such a situation; in this case it is possible that the mean value of the energy $\langle E\rangle_{\psi_{n,\mathbf{R}}}$ is divergent while the mean value $\langle E\rangle_{\psi_{n,\mathbf{Q}_p}}$ is well defined. Now, if the last mean value belongs to **Q**, we have an exact value of the energy with respect to the quantum state ψ. In this case we have constructed a procedure able to give a rational number for the energy when the standard theory would have furnished an infinite value.

We are interested in describing examples of this type. In these cases there is the problem of interpreting statistics for p-adic valued wave functions; an interpretation was proposed in Khrennikov (1992) following a way similar to the standard one. The basis of this interpretation is a p-adic frequency theory of probability [compare to the von Mises (1953) theory of frequency of probability in the real case]. Since relative frequencies are always rational numbers, we define p-adic probabilities as limits of relative frequencies with respect to p-adic topology on **Q**.

2. MATHEMATICAL BASIS

2.1. *p*-Adic Numbers

Let us recall some notions on p-adic numbers. Let **Q** be the field of rational numbers; by means of the standard norm, we can complete it by obtaining the field of real numbers **R**.

A different norm can be introduced on **Q** which is called the p-adic norm; by completing the field **Q** with this norm we get the field of p-adic numbers $\mathbf{Q}_p$.

A famous theorem of number theory (Mahler, 1973) tells us that the field **Q** can be completed only in these two ways.

Let p be a prime positive integer number ($p \neq 1$); for any nonzero rational number $x \in \mathbf{Q}$ there is a unique way to write x as $x = p^{\nu} m/n$. Here m and n are integers which are not divisible by p, while ν is an integer number. This equation is a trivial consequence of the decomposition of x in prime factors.

The p-adic norm is defined as

$$|x|_p = |x^{\nu} m/n|_p = p^{-\nu}, \qquad |0|_p = 0 \tag{1}$$

and satisfies the *strong* triangular inequality

$$|x+y|_p \leq \max(|x|_p, |y|_p). \tag{2}$$

This norm is non-Archimedean (Mahler, 1973).

If we complete $\mathbf{Q}$ with respect to $|\cdot|_p$, we get $\mathbf{Q}_p$, the field of p-adic numbers.

For the convenience of the reader, we recall that any p-adic number can be uniquely written in the form

$$x = \sum_{k=-n}^{\infty} a_k p^k \tag{3}$$

where the numbers a_k are integers, $a_k = 0, 1, \ldots, p-1$. Here the number n is not fixed, but is a function of x. This expression is closely related to the usual decimal expression of a real number.

We write here two inequalities which will be useful in the following:

$$p^{(1-n)/(p-1)} \leq |n!|_p \leq npp^{n/(1-p)}, \qquad |n!|_p \leq |n|_p \leq 1 \tag{4}$$

2.2. p-Adic Gaussian and Lebesgue Distributions

The first definition of p-adic Gaussian distributions was proposed in Khrennikov (1990) on the basis of the theory of p-adic-valued distributions.

p-Adic Gaussian distributions are distributions with quadratic exponent in their Fourier or Laplace transforms. This point is quite complicated; great care has to be taken since there is no possibility to introduce, in p-adic theory, a well-defined, invariant analog of Lebesgue measure (Monna, 1970); if a p-adic map is analytic, then formal integration is possible, but there is no way to extend such a measure to smooth maps. On the contrary, by using the theory of generalized maps (distribution), one can define Laplace and Fourier transforms and give the definition which we stated before.

It can be shown that the Gaussian distribution $v(dx) = \exp(-x^2)\,dx$ can be defined by the requirements

$$\int_{\mathbf{Q}_p} x^{2k+1} e^{-x^2}\,dx = 0 \qquad \text{for all} \quad k = 0, 1, 2, \ldots \tag{5}$$

$$\int_{\mathbf{Q}_p} x^{2k} e^{-x^2}\,dx = \frac{(2k)!}{(k!)^2 4^k} \tag{6}$$

Now, we can extend Gaussian integrals to polynomial and analytical maps by linearity:

$$\int_{\mathbf{Q}_p} f(x) e^{-x^2}\,dx = \sum_{n=0}^{\infty} f_n \int_{\mathbf{Q}_p} x^k e^{-x^2}\,dx, \qquad f_n \in \mathbf{Q}_p \tag{7}$$

If this series is convergent in $\mathbf{Q}_p$, we say that f is summable.

Now we define Lebesgue distribution; for any analytical map $\phi(x)$, we set

$$\int_{\mathbf{Q}_p} \phi(x)\, dx \equiv \int_{\mathbf{Q}_p} (\phi(x)e^{x^2})e^{-x^2}\, dx \tag{8}$$

2.3. Extensions of $\mathbf{Q}_p$ and Hilbert Spaces

First of all, we consider a p-adic analog of complex numbers. We know that the field of complex numbers $\mathbf{C}$ is the quadratic extension of $\mathbf{R}$. In this case we have a very simple algebraic structure because this quadratic extension is, at the same time, the algebraic closure of the field of real numbers. In the p-adic case there is no such simple structure, since there is no unique quadratic extension as in the real case. If $p = 2$, then there are seven different quadratic extensions, and if $p \neq 2$, then there are three different quadratic extensions. All these quadratic extensions are not algebraically closed. The same problem is present for any extensions of finite order. The algebraic closure of $\mathbf{Q}_p$ is constructed as the union of all the algebraic extensions of all orders. Since this algebraic closure is not complete, we must consider the completion of this field. Finally, since this completion is algebraically closed, we define this completion as $\mathbf{C}_p$, the so-called field of complex p-adic numbers (Mahler, 1973).

We introduce now a coordinate Hilbert space $\mathcal{H}$; it is the set of the sequences

$$f = (f_0, f_1, \ldots, f_n, \ldots) \qquad \text{for} \quad f_j \in \mathbf{C}_p \tag{9}$$

such that the expansion

$$|f|^2 = \sum_{n=0}^{\infty} f_n^2 \tag{10}$$

is convergent in $\mathbf{C}_p$. We define also the internal product

$$(f, g) = \sum_{n=0}^{\infty} f_n g_n \tag{11}$$

which is $\mathbf{C}_p$-valued. The main difference with respect to the real case is that we cannot generate a norm by using this scalar product, since $|f|^2$ is, first of all, a $\mathbf{Q}_p$-valued map.

So we define the following norm:

$$\|f\| = \max_{0 \leq n \leq \infty} |f_n|_p \tag{12}$$

which is non-Archimedean:

$$\|f + g\| \leq \max(\|f\|, \|g\|) \tag{13}$$

In this case we have the following analog to the Cauchy equality:

$$|(f, g)|_p \leq \|f\| \cdot \|g\| \tag{14}$$

Let $(E, \|\cdot\|_E)$ be a Banach non-Archimedean $\mathbf{C}_p$-linear space endowed with a symmetric bilinear form $(\cdot, \cdot)_E: E \times E \to \mathbf{C}_p$. If there exists an isomorphism $I: \mathscr{H} \to E$ such that $\|I_f\|_E = \|f\|$ and $(If, Ig)_E = (f, g)$, we say that E is a p-adic Hilbert space (Khrennikov, 1990).

We used this definition to restrict our attention to Hilbert spaces which are isomorphic to coordinate Hilbert spaces, since we have no analog to the classical result stating that any separable Hilbert space is isomorphic to l_2.

A set $\{e_n\}_{n=0}^{\infty}$ of vectors in E is said to be an orthonormal basis of the p-adic Hilbert space E if every $x \in E$ can be uniquely written in the form

$$x = \sum_{n=0}^{\infty} x_n e_n \qquad \text{with} \quad x_n \in \mathbf{C}_p \tag{15}$$

$(e_n, e_m)_E = \delta_{nm}$ and $\|e_n\|_E = 1$.

Remark. If we consider in $\mathscr{H}$ the sequences obtained with a single 1 in the k place, $a_k = (0, 0, \ldots, 1, \ldots, 0, 0)$, then the elements $e_n = Ia_n$ form an orthonormal basis of the p-adic Hilbert space E.

Remark. The dual space E' is not isomorphic to E.

2.4. p-Adic Theory of Probability

Let A be a given observable quantity and let $\Omega = (\alpha_1, \ldots, \alpha_m, \ldots)$ be the set of all possible values assumed by A; let $\mathscr{S}$ be a random experiment measuring A.

If we repeat this experiment N times, we get a sequence $(x_1, \ldots, x_N)$ $(x_j \in \Omega)$ of values obtained from the measure of A. We can compute the relative frequencies after N experiments $\nu_N(\alpha_j) = n(\alpha_j)/N$, where $j = 1, \ldots, N$ and $n(\alpha_j)$ denotes the number of times that the α_j value is obtained in N experiments.

We can ideally consider the limit for $N \to \infty$. In standard theory this limit of elements of $\mathbf{Q}$ is considered with respect to the standard topology of the absolute value, thus yielding a real value probability (von Mises, 1953). In our framework, we consider the limit with respect to the p-adic topology of $\mathbf{Q}$. We so define the *p-adic probability* of the α_j value:

$$P(\alpha_j) = \lim_{N \to \infty} \nu(\alpha_j) \tag{16}$$

This limit does not exist in general; if it exists, following von Mises, we shall say that the random sequence $(x_1, x_2, \ldots, x_N, \ldots)$ is p-adic collective (Khrennikov, 1992).

It can be shown that this probability definition has the additive property, $P(B \cup D) = P(B) + P(D)$ if $B \cap D = \varnothing$, and normalization property, $P(\Omega) = 1$. However, a new unusual property is present: while in the standard theory of probability the limit of the frequency amplitudes, if it exists, is a real number in the segment [0, 1], in the p-adic theory of probability the limit can be any arbitrary p-adic number.

2.5. p-Adic-Valued Quantum Mechanics and Its Statistical Interpretation

The p-adic Hilbert space $(E, \|\cdot\|_E, (\cdot,\cdot)_E)$ is the main object of p-adic-valued quantum mechanics.

We consider vectors $\psi \in E$ as quantum states and $\mathbf{C}_p$-linear operators in E with $\mathbf{Q}_p$ spectrum as observables.

Remark. We have assumed this requirement for the operators describing observables since it is impossible to consider self-adjoint operators; recall that E and E' are not isomorphic.

In this paper we devote our attention to operators with discrete spectrum. Let us consider a physical observable operator $\hat{H}$ and denote by $\{e_i\}_{i=0}^{\infty}$ the set of its eigenvectors:

$$\hat{H}e_i = E_i e_i, \qquad \text{where} \quad E_i \in \mathbf{C}_p \tag{17}$$

We consider the situation in which the set $\{e_i\}_{i=0}^{\infty}$ is a basis for E. At the moment, we have no theorem ensuring that this fact is always true.

Every vector $\psi \in E$ can be expanded in the form

$$\psi = \sum_{i=0}^{\infty} \psi_i e_i, \qquad \text{where} \quad \psi_i \in \mathbf{C}_p \tag{18}$$

We present a p-adic statistical interpretation of the states ψ which are normalized $[|\psi|_E = (\psi, \psi)_E = 1]$ and such that $\psi_i^2 \in \mathbf{Q}_p$ for all i. This last requirement is necessary since it is not possible to introduce a "complex conjugation" in $\mathbf{C}_p$ (W. Schikhof, private communication). The p-adic limit for $N \to \infty$ of the frequency amplitudes $\nu_N(E_i) = n(E_i)/N$ gives the p-adic probability of the value E_i:

$$P(E_i) = \psi_i^2 \qquad \text{for} \quad i = 0, 1, \ldots \tag{19}$$

3. NEW QUANTUM STATES

Let us consider the Hamiltonian

$$\hat{H} = -\frac{d^2}{dx^2} + \lambda^2 x^2 + \lambda, \qquad \text{where} \quad \lambda \in Q, \quad \lambda > 0 \tag{20}$$

The variable x can be regarded as a formal one, and it is possible to realize $\hat{H}$ both in $L_2(\mathbf{R}, dx)$ and in $L_2(\mathbf{Q}_p, dx)$.

Let us consider the analytical map

$$\psi(x) = e^{\lambda x^2/2} \tag{21}$$

Formally, it is an eigenfunction of $\hat{H}$ corresponding to the eigenvalue $E_0 = 0$. Since this function does not belong to $L_2(\mathbf{R}, dx)$, it would be impossible to realize it as a quantum state in usual quantum mechanics; now we propose a probability interpretation of it.

Consider x as a p-adic variable. The function (21) is analytical in a neighborhood of zero. Let us try to normalize it:

$$\begin{aligned} B &= \int_{\mathbf{Q}_p} \psi^2(x)\, dx = \int_{\mathbf{Q}_p} e^{(\lambda+1)x^2} v(dx) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda+1)^n}{n!} \int_{\mathbf{Q}_p} x^{2n} v(dx) \\ &= \sum_{n=0}^{\infty} (\lambda+1)^n \frac{(2n)!}{(n!)^3 4^n} \end{aligned} \tag{22}$$

By using the inequality (4), we can show that this series converges if

$$\lim_{n\to\infty} 2n \frac{(|\lambda+1|_p)^n}{|4^n|_p} p^{(4-p-n)/(1-p)} = 0 \tag{23}$$

For example, if $p = 2$, one can get the estimate $|\lambda + 1|_p < 1/8$ (for instance, $\lambda = 15, 31, \ldots, 13/3, \ldots$).

Let us return to the general case; if equation (22) converges, we can normalize $\psi(x)$ and define $\psi_{n,\mathbf{Q}_p}(x) = \psi/\sqrt{B}$. In this case

$$\|\psi_{n,\mathbf{Q}_p}\|^2_{L_2(\mathbf{Q}_p, dx)} = 1, \qquad \hat{H}\phi_{n,\mathbf{Q}_p} = 0$$

and we can propose a statistical interpretation for the energy $E_0 = 0$ state for this Hamiltonian.

This example can be generalized to a general potential function $V(x)$. Let

$$\hat{H} = -\frac{d^2}{dx^2} + V(x) \tag{24}$$

with

$$V(x) = (\lambda+1)^2 q'^2(x) + (\lambda+1)q''(x) + 2\lambda(\lambda+1)q'(x)x + \lambda^2 x^2 + \lambda \tag{26}$$

and

$$q(x) = a_n x^n + \cdots + a_3 x^3, \qquad \text{where} \quad a_j, \lambda \in Q \tag{26}$$

and $a_n(\lambda + 1) > 0$.

Now let us consider the function

$$f(x) = \exp[(\lambda + 1)q(x) + \lambda x^2/2] \tag{27}$$

Formally, it is an eigenfunction of the Hamiltonian $\hat{H}$ with zero eigenvalue, $E_0 = 0$. Since $a_n(\lambda + 1) > 0$, this map does not belong to $L_2(\mathbf{R}, dx)$ and its normalization is impossible.

Now we try to realize $f(x)$ as a map of p-adic argument. We get

$$A = \int_{\mathbf{Q}_p} f^2(x)\,dx = \int_{\mathbf{Q}_p} e^{(\lambda+1)[2q(x)+x^2]}\nu(dx)$$
$$= \sum_{m=0}^{\infty} \frac{(\lambda+1)^m}{m!} \int_{\mathbf{Q}_p} [2q(x) + x^2]^m \nu(dx) \tag{28}$$

This expansion converges if $|\lambda + 1|_p$ is sufficiently small.

4. *p*-ADIC RENORMALIZATION OF EIGENFUNCTIONS

In this section we study a case in which the wave function ψ belongs to both spaces $L_2(\mathbf{R}, dx)$ and $L_2(\mathbf{Q}_p, dx)$, but the computation of the expectation value of the energy is possible only in the p-adic picture.

Let us consider the Hamiltonian of the harmonic oscillator:

$$\hat{H} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \tag{29}$$

If $\{h_n(x)\}_{n=0}^{\infty}$ is a system of Hermitian polynomials

$$h_n(x) = e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{30}$$

then

$$\hat{H} e_n(x) = \left(n + \frac{1}{2}\right) e_n(x) \tag{31}$$

where $e_n(x) = h_n(x) \exp(-x^2/2)$.

If we calculate

$$\int_{-\infty}^{+\infty} e_n^2(x)\,dx = \sqrt{\pi}\,2^n n! \tag{32}$$

we see that the functions

$$\bar{e}_n(x) = e_n(x) \frac{1}{\pi^{1/4}(2^n n!)^{1/2}} \tag{33}$$

are normalized with respect to the $L_2(\mathbf{R}, dx)$ norm. Consider now an arbitrary function

$$u(x) = \sum_{i=0}^{\infty} u_i \bar{e}_i(x) \in L_2(\mathbf{R}, dx) \tag{34}$$

which we suppose normalized. Compute the mean value of the energy; it is

$$\langle E \rangle_u = (\hat{H}u, u) = \sum_{i=0}^{\infty} \left(i + \frac{1}{2}\right) |u_i|^2 \tag{35}$$

Is this series convergent or not? It depends.

Example. Suppose that p is any fixed prime number. Let $u_i \neq 0$ iff $i = (p+1)^{2k}$ $(k = 0, 1, 2, \ldots)$ and $u_i = p^k/(p+1)^k$ in this case:

$$u(x) = \sum_{i=0}^{\infty} \left(\frac{p}{p+1}\right)^k \bar{e}_i(x) \tag{36}$$

Since $\|u\|^2 = (p+1)^2/(2p+1)$, we can normalize u by putting

$$u_n(x) = \left[\frac{2p+1}{(p+1)^2}\right]^{1/2} \sum_{i=0}^{\infty} \left(\frac{p}{p+1}\right)^k \bar{e}_i(x) \tag{37}$$

Let us consider the series (35) in this case:

$$\langle E \rangle_{u_n} = \frac{2p+1}{(p+1)^2} \sum_{k=0}^{\infty} p^{2k} + \frac{1}{2} \tag{38}$$

the first term is not convergent. Thus, it would be impossible to compute the mean value of the energy with respect to this normalized state.

Now we shall use p-adic theory to try to resolve this problem. We use the same expression for the functions $e_n(x)$ by regarding x as a p-adic variable. For the normalization we use the integral

$$\int_{\mathbf{Q}_p} e_n^2(x)\, dx = 2^n n! \tag{39}$$

and we set

$$\bar{e}_{n,\mathbf{Q}_p}(x) = e_n(x) \frac{1}{(2^n n!)^{1/2}} \tag{40}$$

Consider the function formally obtained from $u(x)$ of equation (36) by regarding x as a p-adic variable and by substituting $\bar{e}_n(x)$ with $\bar{e}_{n,\mathbf{Q}_p}(x)$.

Also in this case, we can compute the norm of $u(x)$ in $L_2(\mathbf{Q}_p, dx)$ and normalize it by putting again $u_n(x) = u(x)[(2p+1)/(p+1)^2]^{1/2}$. Notice that the number $[(2p+1)/(p+1)^2]^{1/2}$ is p-adic.

Now we compute

$$\begin{aligned}\langle E\rangle_{u_n} &= \frac{1}{2} + \frac{2p+1}{(p+1)^2}\sum_{k=0}^{\infty} p^{2k} \\ &= \frac{1}{2} + \frac{2p+1}{(p+1)^2}\frac{1}{1-p^2} \\ &= \frac{p^4+2p^3-6p-3}{2(p^2-1)(p+1)^2}\end{aligned} \tag{41}$$

This energy value is correctly positive for every choice of p.

We have shown that the theory of p-adic numbers in some cases is able to furnish a finite rational value of observable quantities in cases in which standard real (or complex) quantum mechanics would give rise to an infinite value.

One can try to consider similar situations in quantum field theory. At the moment we have no definitive results in this direction, but we think that p-adic analysis is useful in perturbation theories for the computation of divergent integrals.

This suggestion is also supported by the results of Smirnov (1991), where, by considering complex-valued fields of p-adic argument, it was shown that, in the Feynman integrals, only logarithmic divergences are involved. See also Smirnov (1992).

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